On an Extension of Abel-Gontscharoff's Expansion Formula and Its Application to Multivariate Interpolation

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Abstract

IHere presented is constructive generalization of Abel-Gontscharoff's series expansion to higher dimensions. Moreover, a constructive application to a problem of multivariate interpolation is investigated. Another approach of Kergin type multivariate interpolation by using Abel-Gontscharoff-Gould polynomial is also discussed.

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1 Introduction

Throughout we should adopt various notations in the multiple-index system.

(i) Given $\nu \in \mathbb{N}^s$ with $\nu = (\nu_1, \nu_2, \cdots, \nu_s)$. We denote

$$|\nu| := \sum_{i=1}^{s} \nu_i, \qquad \nu! := \nu_1!\nu_2! \cdots \nu_s!.$$

If $k \in \mathbb{N}^s$ such that $k \leq \nu$ (i.e., $k_i \leq \nu_i$, $1 \leq i \leq s$), then we denote

$$\binom{\nu}{k} := \binom{\nu_1}{k_1}\binom{\nu_2}{k_2} \cdots \binom{\nu_s}{k_s} = \frac{\nu!}{k!(\nu - k)!},$$

where $\nu - k = (\nu_1 - k_1, \nu_2 - k_2, \cdots, \nu_s - k_s)$.

(ii) Given $\nu \in \mathbb{N}^s$ and $x \in \mathbb{R}^s$ (or $\mathbb{C}^s$) with $x = (x_1, x_2, \cdots, x_s)$. We denote

$$x^\nu := x_1^{\nu_1} x_2^{\nu_2} \cdots x_s^{\nu_s}.$$

(iii) Given $x, \alpha \in \mathbb{R}^s$ (or $\mathbb{C}^s$). We denote

$$\partial^\nu f(x) := \frac{\partial^{|\nu|} f(x)}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \cdots \partial x_s^{\nu_s}}, \;\; \partial^\nu f(\alpha) = \partial^\nu f(\alpha) = \partial^\nu f(x)\big|_{x=\alpha},$$

and if $\nu = \mathbf{0} = (0, \cdots, 0)$ we denote $\partial^{\mathbf{0}} f(\alpha) = f(\alpha)$.

(iv) For $k \in \mathbb{N}^s$ we define $\alpha_k \in \mathbb{R}^s$ as a multiple sequence of dimension s, i.e.,

$$\alpha_k = (\alpha_k^{(1)}, \alpha_k^{(2)}, \cdots, \alpha_{k_s}^{(s)}, \quad (k \in \mathbb{N}^s).$$

Also, we write $\infty = (\infty, \cdots, \infty)$.

The multivariate Taylor-Maclaurin expansion at $\mathbf{0}$ is given by

$$f(x) \;=\; \sum_{\nu \geq \mathbf{0}} \frac{1}{\nu!} \partial^\nu f(\mathbf{0}) x^\nu \tag{1.1}$$

$$f(x) \;=\; \sum_{|\nu| \leq r} \frac{1}{\nu!} \partial^\nu f(\mathbf{0}) x^\nu + \sum^{|\nu|=r+1} \frac{1}{\nu 1} \partial^\nu f(\theta x) x^\nu, , \tag{1.2}$$

where $0 < \theta < 1$ and $\theta x = (\theta x_1, \theta x_2, \cdots, \theta x_s)$, $r \geq 1$.

For the simple case $s = 1$, let $n, k \in \mathbb{N}$ and $x \in \mathbb{C}$. Given $\beta_k \in \mathbb{C}$. It is known that Gould's algebraic identity takes the form

$$\sum_{k=0}^{n}\binom{n}{k}c(k)(x-\beta_k)^{n-k}=x^n, \tag{1.3}$$

in which $c(0)=1$, and $c(k)\equiv c(k;\beta)\equiv c(k;\beta_0,\beta_1,\cdots,\beta_{k-1})$ is a kind of homogeneous polynomial, claaed the Abel-Gontscharoff-Gould polynomial, of degree k in β_1, $\beta_2,\cdots,\beta_{k-1}$. For more details about $c(k)$, see Gould [1], Hsu [2], and He-Hsu-Shiue [3].

Let *Gamma* $\equiv(\Gamma,+,\cdot)$ be the commutative ring of formal power series over $\mathbb{C}^s$ (or $\mathbb{R}^s$), and let $\alpha_k\equiv\left(\alpha^{(1)}_{k_1},\alpha^{(2)}_{k_2},\cdots,\alpha^{(s)}_{k_s}\right)$ $(k\in\mathbb{N}^s)$ be a given multiplesequence. Then, what a basic result to be proved in this paper (cf. Section 3) is the following: For any $f\in\Gamma$ we have a formal series expansion of the form

$$f(x)=\sum_{k\geq\mathbf{0}}\frac{\partial^k f(d_k)}{k!}C(k;x-\alpha_0,x-\alpha_1,\cdots,x-\alpha_{k-1}), \tag{1.4}$$

where $C(k;x-\alpha_0,x-\alpha_1,\cdots,x-\alpha_{k-1})$ is given by

$$\begin{aligned} & C(k;x-\alpha_0,x-\alpha_1,\cdots,x-\alpha_{k-1}) \\ = \; & \Pi_{i=1}^{s}c\left(k;x_i-\alpha_0^{(i)},x_i-\alpha_1^{(i)},\cdots,x_i-\alpha^{(i)}_{k_i-1}\right) \end{aligned} \tag{1.5}$$

Evidently, the classical Abel-Gontscharoff series expansion (i.e., Abel-Gontscharoff interpolation series) is a particular case of (1.4) and (1.5) with $s=1$. Also we shall show that a multivariate polynomial $\Phi_r(x)\equiv\Phi_r(f;x)\in\pi_r^s$ (the set of all polynomials of degree $\leq k$ in s variables) of the form (with $r\geq 1$)

$$\Phi_r(f;x)=\sum_{|k|\leq r}\frac{\partial^k f(\alpha_k)}{k!}C(k;x-\alpha_0,x-\alpha_1,\cdots,x-\alpha_{k-1}) \tag{1.6}$$

just solves a general problem for multivariate interpolation. This will be discussed latter in Section 3.

There are still two other types of multivariate generalization of Abel-Gontscharoff interpolation series. One of them has been mentioned briefly in Remark 8 of our paper [3]. The second one has been given in a recent paper by one of the authors in [4], in which the classic Abel-Gontscharoff interpolation is extended to the multivariate Kergin interpolation by using a differential operator generated from Abel-Gontscharoff-Gould polynomial. A summary of the results will be sketched in Section 4. In the last section, Section 5, we give an algorithm for computing the Abel-Gontscharoff-Gould polynomial defined by (1.5).

2 Lemmas Needed

Lemma 2.1 *(A multivariate form of Gould's identity) Let* $x \in \mathbf{C}^s$ *and* $\nu, k \in \mathbf{N}^s$*, and denote* $\beta^{(i)} \equiv \left(\beta_0^{(i)}, \beta_1^{(i)}, \cdots, \beta_{k_i-1}^{(i)}\right)$*,* $(i = 1, 2, \cdots)$*. We have an algebraic identity of the form*

$$x^\nu = \sum_{k \le \nu} \binom{\nu}{k} \Pi_{i=1}^s c\left(k_i; \beta^{(i)}\right) \left(x_i - \beta_{k_i}^{(i)}\right)^{\nu_i - k_i}. \tag{2.7}$$

Proof. Application of (1.3) to ach of the s factors of $x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots x_s^{\nu_s}$ yields the result expression (2.7).

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Lemma 2.2 *Let* $\nu \in \mathbb{N}^s$*,* $x \in \mathbb{C}^s$*, and* $\alpha_k \equiv \left(\alpha_{k_1}^{(1)}, \alpha_{k_2}^{(2)}, \cdots, \alpha_{k_s}^{(s)}\right)$ *with* $k \in \mathbb{N}^s$*. Then we have*

$$\partial^\nu \Pi_{i=1}^s c\left(k_i; x_i - \alpha_0^{(i)}, x_i - \alpha_1^{(i)}, \cdots, x_i - \alpha_{k_i-1}^{(i)}\right)\Big|_{x=\alpha_\nu} = \begin{cases} 0 & \text{if } \nu \ne k, \\ k! & \text{if } \nu = k. \end{cases} \tag{2.8}$$

Proof. This follows easily from a repeated application of the equation (2.8) of our paper [3]. Indeed, in accordance with Proposition 2.1 of [3] we see that the left-hand side of (2.8) can be re-written in the form

$$\begin{aligned} & \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \cdots \partial x_s^{\nu_s}} \Pi_{i=1}^s Q_{k_i}(x_i)\Bigg|_{x=\alpha_\nu} = \Pi_{i=1}^s Q_{k_i}^{(\nu_i)}\left(\alpha_{\nu_i}^{(i)}\right) \\ = \quad & \Pi_{i=1}^s \nu_i! \delta_{\nu_i k_i} = \begin{cases} 0 & \text{if } \nu \ne k, \\ k! & \text{if } \nu = k, \end{cases} \end{aligned}$$

wherein $\delta_{\cdot,\cdot}$ is the Kronecker symbol.

■

3 Main Theorems and Remarks

Theorem 3.1 *Let $\{\alpha_k\}$ be a given s-multiple sequence with $\alpha_k \equiv \left(\alpha^{(1)}_{k_1}, \alpha^{(2)}_{k_2}, \cdots, \alpha^{(s)}_{k_s}\right) \in \mathbb{C}^s$ (or $\mathbb{R}^s$) and $k \in bN^s$. Then for any $f \in \Gamma\left(bC^s\right)$ we have the formal series expansion formula*

$$f(x) = \sum_{k\geq \mathbf{0}} \frac{\partial^k f(\alpha_k)}{k!} \Pi_{i=1}^s c\left(k_i; x_i - \alpha^{(i)}_0, x_i - \alpha^{(i)}_1, \cdots, x_i - \alpha^{(i)}_{k_i-1}\right), \tag{3.9}$$

where $x \in \mathbb{C}^s$ (or $\mathbb{R}^s$), and $c\left(k_i; x_i - \alpha^{(i)}_0, x_i - \alpha^{(i)}_1, \cdots, x_i - \alpha^{(i)}_{k_i-1}\right)$ are Abel-Gontscharoff-Gould polynomials with degrees $k_i \in \mathbb{N}$.

Proof. For brevity let us denote

$$\beta^{(i)} \equiv \left(\beta^{(i)}_0, \beta^{(i)}_1, \cdots, \beta^{(i)}_{k_i-1}\right), \quad C(k;\beta) \equiv \Pi_{i=1}^s c(k_i; \beta^{(i)}).$$

Also we shall make the substitutions $\beta^{(i)}_j = x_i - \beta^{(i)}_j$, $(i, j \in \mathbb{N})$. Then using Lemma 1.1 and the multivariate Taylor expansion, we see that $f(x)$ can be formally expanded as follows.

$$\begin{aligned} f(x) &= \sum_{\nu\geq \mathbf{0}} \frac{\partial^\nu f(\mathbf{0})}{\nu!} x^\nu \\ &= \sum_{\nu\geq \mathbf{0}} \frac{\partial^\nu f(\mathbf{0})}{\nu!} \sum_{k\leq \nu} \binom{\nu}{k} \Pi_{i=1}^s c(k_i; \beta^{(i)} \left(x_i - \beta^{(i)}_{k_i}\right)^{\nu_i - k_i} \\ &= \sum_{k\geq \mathbf{0}} \frac{C(k;\beta)}{k!} \sum_{\nu\geq k} \frac{\partial^\nu f(\mathbf{0})}{(\nu-k)!} \left(x_i - \beta^{(i)}_{k_i}\right)^{\nu_i - k_i} \end{aligned}$$

By substituting $\mu = \nu - k = (\mu_1, \mu_2, \cdots, \mu_s)$ into the right-hand side of the last equation we can write it as

$$\begin{aligned}
&\sum_{k\geq\mathbf{0}}\frac{C(k;\beta)}{k!}\sum_{\mu\geq k}\frac{\partial^{\mu+k}f(\mathbf{0})}{\mu!}\left(x_i-\beta^{(i)}_{k_i}\right)^{\mu_i}\\
=&\sum_{k\geq\mathbf{0}}\frac{C(k;\beta)}{k!}\sum_{\mu\geq k}\frac{1}{\mu!}\partial^{\mu}\left(\partial^k f(\mathbf{0})\left(x_i-\beta^{(i)}_{k_i}\right)^{\mu_i}\right.\\
=&\sum_{k\geq\mathbf{0}}\frac{C(k;\beta)}{k!}\partial^k f\left(x_1-\beta^{(1)}_{k_1},x_2-\beta^{(2)}_{k_2},\cdots,x_s-\beta^{(s)}_{k_s}\right)\\
=&\sum_{k\geq\mathbf{0}}\frac{1}{k!}\partial^k f\left(\alpha^{(1)}_{k_1},\alpha^{(2)}_{k_2},\cdots,\alpha^{(s)}_{k_s}\right)\\
&\times\Pi^s_{i=1}c\left(k_i;x_i-\alpha^{(i)}_0,x_i-\alpha^{(i)}_1,\cdots,x_i-\alpha^{(i)}_{k_i-1}\right),
\end{aligned}$$

which is just the right-hand side of Eq. (3.9).

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For the case $\alpha_k\equiv(a_1,a_2,\cdots,a_s)\equiv a\in\mathbb{C}^s$, so that $\alpha^{(i)}_j=a_i$, we see that (3.9) implies the multivariate Taylor expansion as a consequence, namely

$$f(x)=\sum_{k\geq\mathbf{0}}\frac{\partial^k f(kt)}{k!}\Pi^s_{i=1}x_i(x_i-k_it_i)^{k_i-1}, \tag{3.10}$$

where $k\in\mathbb{N}^s$, $x,t\in\mathbb{C}^s$ and $kt\equiv(k_1t_1,k_2t_2,\cdots,k_st_s)$.

Theorem 3.2 *Given* $\alpha_k\equiv\left(\alpha^{(1)}_{k_1},\alpha^{(2)}_{k_2},\cdots,\alpha^{(s)}_{k_s}\right)\in\mathbb{C}^s$ *with* $k\in\mathbb{N}^s$, *let* $f(x)\in\Gamma$ *(over* $\mathbb{C}^s$*). Then the* s*-variate Abel-Gontscharoff polynomial of degree* r $(r\geq 1)$ *given by*

$$\begin{aligned}
&\Phi_r(x)=\Phi_r(f;x)\\
=&\sum_{|k|\leq r}\frac{\partial^k f(\alpha_k)}{k!}\Pi^s_{i=1}c\left(k_i;x_i-\alpha^{(i)}_0,x_i-\alpha^{(i)}_1,\cdots,x_i-\alpha^{(i)}_{k_i-1}\right)
\end{aligned}\tag{3.11}$$

satisfies the interpolation conditions

$$\left\{\begin{array}{ll}\Phi_r(\alpha_0)=f(\alpha_0), & \left(\alpha_0\equiv\left(\alpha^{(1)}_0,\alpha^{(2)}_0,\cdots,\alpha^{(s)}_0\right)\right),\\ \partial^\nu\Phi_r(x)|_{x=\alpha_\nu}=\partial^\nu f(\alpha_\nu), & 1\leq|\nu|\leq r.\end{array}\right. \tag{3.12}$$

Proof. In the first place, notice that $c(j;\beta_0,\beta_1,\cdots,\beta_{j-1})=0$ for $j\in\mathbb{N}$, $j\geq 1$ and $\beta_0=0$. Thus we have

$$\Pi_{i=1}^{s}c\left(k_i;x_i-\alpha_0^{(i)},x_i-\alpha_1^{(i)},\cdots,x_i-\alpha_{k_i-1}^{(i)}\right)\Big|_{x=\alpha_0}=0,\quad (|k|\geq 1).$$

Moreover, $c(0;\beta)=1$. So it follows that

$$\Phi_r(\alpha_0)=\Phi_r(f;x)|_{x=\alpha_0}=\partial^0 f(\alpha_0)=f(\alpha_0).$$

Furthermore, for $|\nu|\geq 1$ we have by using Lemma 2.2

$$\begin{aligned}
&\partial^\nu\Phi_r(x)|_{x=\alpha_\nu}\\
=\ &\sum_{|k|\leq r}\frac{\partial^k f(\alpha_k)}{k!}\partial^\nu\Pi_{i=1}^{s}c\left(k_i;x_i-\alpha_0^{(i)},x_i-\alpha_1^{(i)},\cdots,x_i-\alpha_{k_i-1}^{(i)}\right)\Bigg|_{x=\alpha_\nu}\\
=\ &\frac{\partial^k f(\alpha_k)}{k!}\left\{\begin{array}{ll}0 & (k\neq\nu)\\ k! & (k=\nu)\end{array}\right\}=\partial^\nu f(\alpha_\nu).
\end{aligned}$$

Hence Theorem 3.2 is proved.

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With the notation given by (1.5) the polynomial defined by (3.11) may be written in a more compact from, namely

$$\Phi_r(f;x)=\sum_{|k|\leq r}\frac{\partial^k f(\alpha_k)}{k!}C(k;x-\alpha_0,x-\alpha_1,\cdots,x-\alpha_{k-1}),\qquad(3.13)$$

where $x\in\mathbb{C}^s$, $k\in\mathbb{N}^s$, and $\{\alpha_k\}$ is a given sequence with $\alpha_k\in\mathbb{C}^s$. Certainly, $\Phi_r(f;x)$ may be called the s-variate Abel-Gontscharoff interpolation polynomial of degree r.

The difference $\rho_r(f;x)=f(x)-\Phi_r(f;x)$ is called a remainder of the expression (3.9). From the viewpoint of numerical analysis, it may be of interest to find some useful expression for $\rho_r(f;x)$ ($r\geq 1$). In the following, we present an expression of $\rho_r(f;x)$ by using Lemma 2.1.

Theorem 3.3 *Given* $\alpha_k\equiv\left(\alpha_{k_1}^{(1)},\alpha_{k_2}^{(2)},\cdots,\alpha_{k_s}^{(s)}\right)\in\mathbb{C}^s$ *with* $k\in\mathbb{N}^s$, *let* $f(x)\in\Gamma$ *(over* $\mathbb{C}^s$*). Then the* s*-variate Abel-Gontscharoff polynomial of degree* r *(*$r\geq 1$*) given by (3.11) has remainder*

$$\rho_r(f;x)=\sum_{|k|\le r+1}\frac{\partial^k f\left((1+\theta)x_1+\alpha^{(1)}_{k_1},\cdots,(1+\theta)x_s+\alpha^{(s)}_{k_s}\right)}{k!}$$
$$\times\Pi_{i=1}^s c\left(k_i;x_i-\alpha^{(i)}_0,x_i-\alpha^{(i)}_1,\cdots,x_i-\alpha^{(i)}_{k_i-1}\right), \qquad (3.14)$$

where $0<\theta<1$ *and* $\theta x=(\theta x_1,\theta x_2,\cdots,\theta x_s)$.

Proof. Comparing Eqs. (1.1)-(1.2) and (3.9)-(3.11) and using Lemma 2.1, we obtain

$$\rho_r(f;x)=\sum_{|\nu|=r+1}\frac{1}{\nu!}\partial^\nu f(\theta x)x^\nu$$
$$=\sum_{|\nu|=r+1}\frac{1}{\nu!}\partial^\nu f(\theta x)\sum_{k\le\nu}\binom{\nu}{k}\Pi_{i=1}^s c(k_i;\beta^{(i)})\left(x_i-\beta^{(i)}_{k_i}\right)^{\nu_i-k_i}$$
$$=\sum_{|k|\le r+1}\frac{C(k;\beta)}{k!}\sum_{\nu\ge k}\frac{\partial^\nu f(\theta x)}{(\nu-k)!}\Pi_{i=1}^s\left(x_i-\beta^{(i)}_{k_i}\right)^{\nu_i-k_i}.$$

Substituting $\mu=\nu-k$ into the right-hand side of the above equation yields

$$\rho_r(f;x)$$
$$=\sum_{|k|\le r+1}\frac{C(k;\beta)}{k!}\sum_{\mu\ge\mathbf{0}}\frac{\partial^{\mu+k}f(\theta x)}{\mu!}\Pi_{i=1}^s\left(x_i-\beta^{(i)}_{k_i}\right)^{\mu_i}$$
$$=\sum_{|k|\le r+1}\frac{C(k;\beta)}{k!}\sum_{\mu\ge\mathbf{0}}\frac{1}{\mu!}\partial^\mu\left(\partial^k f(\theta x)\right)\Pi_{i=1}^s\left(x_i-\beta^{(i)}_{k_i}\right)^{\mu_i}$$
$$=\sum_{|k|\le r+1}\frac{C(k;\beta)}{k!}\partial^k\left((1+\theta)x_1-\beta^{(1)}_{k_1},\cdots,(1+\theta)x_s-\beta^{(s)}_{k_s}\right)$$
$$=\sum_{|k|\le r+1}\partial^k\left((1+\theta)x_1-\beta^{(1)}_{k_1},\cdots,(1+\theta)x_s-\beta^{(s)}_{k_s}\right)$$
$$\times\frac{1}{k!}\Pi_{i=1}^s c\left(k_i;x_i-\alpha^{(i)}_0,x_i-\alpha^{(i)}_1,\cdots,x_i-\alpha^{(i)}_{k_i-1}\right).$$

This completes the proof of the theorem.

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Obviously, the remainder in (1.2) is a special case of expression (3.14).

4 Kergin type interpolation

5 Algorithm on the computation of Abel-Gontscharoff-Gould polynomials

In order to make (3.13) (or (3.9) really available, it needs to devise some algorithms for computing the Abel-Gontscharoff-Gould polynomials defined by (1.5). In this section we shall discuss the computation of the polynomials.

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