A symbolic operator approach to several summation formulas for power series II

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Abstract

Here expounded is a kind of symbolic operator method that can be used to construct many transformation formulas and summation formulas for various types of power series including some old ones and more new ones.

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1 Introduction - sketch of an operator method

As is well-known, the closed form representation of series has been studied extensively. See, for examples, Comtet [1], Jordan [12], Egorychev [2], Roman-Rota [14], Sofo [15], Wilf [17], Petkovšek-Wilf-Zeilberger's book [13], and the authors' recent work [8]. This paper is a sequel to the authors with Torney paper [6]. The object of this paper is to make use of the classical operators Δ (difference), E (shift), and D (derivative) to construct a method for the summation of power series that appears to have a certain wide scope of applications.

An important tool used in the Calculus of Finite Differences and in Combinatorial Analysis are the operators E, Δ, and D defined by the following relations.

$$Ef(t) = f(t+1), \quad \Delta f(t) = f(t+1) - f(t), \quad Df(t) = \frac{d}{dt} f(t).$$

Powers of these operators are defined in the usual way. In particular for any real numbers x, one may define $E^x f(t) = f(t+x)$. Also, the number 1 is defined as an identity operator, viz. $1f(t) \equiv f(t)$. It is easy to verify that these operators satisfy the formal relations (*cf.* [12])

$$E = 1 + \Delta = e^D, \quad \Delta = E - 1 = e^D - 1, \quad D = \log(1+\Delta).$$

Note that $E^k f(0) = \left[E^k f(t)\right]_{t=0} = f(k)$, so that $(xE)^k f(0) = f(k)x^k$. This means that $(xE)^k$ with x as a parameter may be used to generate a general term of the series $\sum_{k=0}^{\infty} f(k)x^k$. Now suppose that $\Phi(t)$ is an analytic function of t or a formal power series in t, say

$$\Phi(t) = \sum_{k=0}^{\infty} c_k t^k, \qquad c_k = [t^k]\Phi(t), \tag{1.1}$$

where c_k can be either real or complex numbers. Then, formally we have a sum of general form

$$\Phi(xE)f(0) = \sum_{k=0}^{\infty} c_k f(k) x^k. \tag{1.2}$$

The operator $\Phi(xE) = \Phi(x + x\Delta) = \Phi(xe^D)$ can be expressed as some power series involving operators Δ^k or D^k's. Then it may be possible to compute the right-hand side of (1.2) by means of operator-series in Δ^k or D^k's. This idea could be readily applied to various elementary functions $\Phi(t)$. Indeed, if we take $\Phi(t)$ to be any of the following functions

$$\begin{array}{lll} (i)\ (1+t)^{\alpha}, & (ii)\ (1-t)^{-\alpha-1}, & (iii)\ e^t \\ (iv)\ -\log(1-t), & (v)\ \sin t, & (vi)\ \cosh t, \end{array} \tag{1.3}$$

etc., thus, using suitable expressions of $\Phi(xE) = \Phi(x + x\Delta) = \Phi(xe^D)$ in terms of Δ^k or D^k, we can obtain various transformation formulas as well as summation formulas for the series of the form (1.2).

Our results that will be presented in this paper are a significant improvement of our previous work with Torney shown in [6], in which the main result is a special case of Theorem 3.1.

Remark 1.1 Obviously, $\Phi(t)$ is not limited to the functions shown as in (1.3). For instance, we may choose (*cf.* [10])

$$\Phi(t) = (1 - mzt + yt^m)^{-\lambda} = \sum_{k=0}^{\infty} P_k(m, z, y, \lambda)t^k, \tag{1.4}$$

the generating functions (GFs) of the so-called Gegenbauer-Humbert-type polynomials. As specific cases of (1.4), we consider $P_k(m, z, y, \lambda)$ as follows

$$\begin{aligned} &P_k(2, z, 1, 1) = U_k(z),\ \textit{Chebyshev 2nd kind polynomial}, \\ &P_k(2, z, 1, 1/2) = \psi_k(z),\ \textit{Legendre polynomial}, \\ &P_k(2, z, -1, 1) = P_{k+1}(z),\ \textit{Pell polynomial}, \\ &P_k(2, z/2, -1, 1) = F_{k+1}(z),\ \textit{Fibonacci polynomial}, \\ &P_k(2, z/2, 2, 1) = \Phi_{k+1}(z),\ \textit{Fermat 1st kind polynomial}, \end{aligned}$$

where $F_{k+1} = F_{k+1}(1)$ is the Fibonacci number.

The expansion (1.4) is a special case of the generalized Humbert polynomials studied by Gould in [3], in which a generalized Humbert polynomial $P_n(m, x, y, p, C)$ is defined by means of

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} t^n P_n(m, x, y, p, C),$$

where m is an integer ≥ 1 and the other parameters are unrestricted. In that paper, Gould first obtained some recurrences satisfied by the P_n and then gives a formula for ${D_x}^k P_{n+k}$ that generalizes a formula of Catalan for the kth derivative of the Legendre polynomial. He also showed that if the function $f(x,t)$ satisfies $(tD_t)f(x,t) = (x - yt^{m-1})D_x f(x,t)$, then

$$(tD_t)^r f(x,t) = \sum_{j=1}^{r} {Q_j}^r(m,x,y,t){D_x}^j f(x,t) \quad (r \geq 1),$$

where

$$p!(-mt)^p {Q_p}^r(m,x,y,t) = \sum_{n=0}^{mp} n^r t^n P_n(m,x,y,p,mxt - yt^m) \quad (1 \leq p \leq r).$$

Some notations and an extension of Eulerian fractions will be given in next section. Two lists of transformation and summation formulas will be displayed in the latter Sections 3, and many illustrative examples will be given in Section 4.

2 An extension of Eulerian fractions

It is well-known that the Eulerian fraction is a powerful tool in the study of the Eulerian polynomial, Euler function and its generalization, Jordan function (*cf.* [1]).

The classical Eulerian fraction, $\alpha_m(x)$, can be expressed in the form

$$\alpha_m(x) = \frac{A_m(x)}{(1-x)^{m+1}} \quad (x \neq -1), \tag{2.1}$$

where $A_m(x)$ is the mth degree Eulerian polynomial of the form

$$A_m(x) = \sum_{j=0}^{m} j! \left\{ \begin{matrix} m \\ j \end{matrix} \right\} x^j (1-x)^{m-j}, \tag{2.2}$$

$\left\{ \begin{matrix} m \\ j \end{matrix} \right\}$ being Stirling numbers of the second kind, i.e., $j! \left\{ \begin{matrix} m \\ j \end{matrix} \right\} = [\Delta^j t^m]_{t=0}$. Evidently $\alpha_m(x)$ can be written in the form (*cf.* [6])

$$\alpha_m(x) = \sum_{j=0}^{m} \frac{j! \left\{ \begin{matrix} m \\ j \end{matrix} \right\} x^j}{(1-x)^{j+1}}.$$

In order to express some new formulas for certain general types of power series, we need to introduce an extension of Euler fraction associated with an infinitely differentiable function $g(x)$ defined as

$$A_m(x, g(x)) := \sum_{j=0}^{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} g^{(j)}(x) x^j, \tag{2.3}$$

where $g^{(j)}(x)$ is the j-th derivative of $g(x)$. Obviously, $\alpha_m(x)$ defined by (2.1) can be presented as

$$\alpha_m(x) = A_m(x, (1-x)^{-1}).$$

From (2.3), two kinds of generalized Eulerian fractions in terms of $g(x) = (1+x)^\alpha$ and $g(x) = (1-x)^{-\alpha-1}$, with real number α as a parameter, can be introduced respectively, namely

$$\begin{aligned} A_m(x, \alpha) &\equiv A_m\left(x, (1+x)^\alpha\right) = \frac{A_m(x)_\alpha}{(1+x)^{m-\alpha}} \\ &= \sum_{j=0}^{m} \binom{\alpha}{j} \frac{j! \left\{ \begin{matrix} m \\ j \end{matrix} \right\} x^j}{(1+x)^{j-\alpha}}, \quad (x \neq -1), \qquad (2.4) \\ \tilde{A}_m(x, \alpha) &\equiv A_m\left(x, (1-x)^{-\alpha-1}\right) = \frac{\tilde{A}_m(x)_\alpha}{(1-x)^{\alpha+m+1}} \\ &= \sum_{j=0}^{m} \binom{\alpha+j}{j} \frac{j! \left\{ \begin{matrix} m \\ j \end{matrix} \right\} x^j}{(1-x)^{\alpha+j+1}}, \quad (x \neq 1). \qquad (2.5) \end{aligned}$$

These may be called, respectively, the 1st kind and 2nd kind of generalized Eulerian fractions. Correspondingly $A_m(x, \alpha)$ and $\tilde{A}_m(x, \alpha)$ are called the mth degree generalized Eulerian polynomials, having explicit expressions as follows:

$$\begin{aligned} A_m(x,\alpha) &= \sum_{j=0}^{m} \binom{\alpha}{j} j! \left\{ \begin{array}{c} m \\ j \end{array} \right\} x^j (1+x)^{m-j} && (2.6) \\ \tilde{A}_m(x,\alpha) &= \sum_{j=0}^{m} \binom{\alpha+j}{j} j! \left\{ \begin{array}{c} m \\ j \end{array} \right\} x^j (1-x)^{m-j}. && (2.7) \end{aligned}$$

As easily seen, $\tilde{A}_m(x,0) = \alpha_m(x) = A_m(-x,-1)$.

3 Series-transformation formulas

All formulas presented in this section are formal identities in which we always assume that $x \neq 1$ or $x \neq -1$ according as $(1-x)^{-1}$ or $(1+x)^{-1}$ appears in the formulas.

Theorem 3.1 *Let $\{f(k)\}$ be a given sequence of numbers (real or complex), and let $g(t)$ and $h(t)$ be infinitely differentiable on $[0,\infty)$. Then we have formally*

$$\sum_{k=0}^{\infty} f(k) g^{(k)}(0) \frac{x^k}{k!} = \sum_{k=0}^{\infty} \Delta^k f(0) g^{(k)}(x) \frac{x^k}{k!} \qquad (3.1)$$

$$\sum_{k=0}^{\infty} h(k) g^{(k)}(0) \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} h^{(k)}(0) A_k(x, g(x)), \qquad (3.2)$$

where $A_m(x, g(x))$ is an extension of Euler fraction in terms of $g(x)$ defined as in (2.3).

Proof. To prove (3.1), we apply the operator $g(xE)$ to $f(t)$ at $t=0$, where E is the shift operator.

$$g(xE)f(t)|_{t=0} = \sum_{k=0}^{\infty} \frac{1}{k!} g^{(k)}(0)\, (xE)^k f(t)\big|_{t=0} = \sum_{k=0}^{\infty} f(k) g^{(k)}(0) \frac{x^k}{k!}.$$

On the other hand, we have

$$
\begin{aligned}
&g(xE)f(t)|_{t=0} = g(x + x\Delta)f(t)|_{t=0} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} g^{(k)}(x)\ (x\Delta)^k f(t)\big|_{t=0} = \sum_{k=0}^{\infty} \Delta^k f(0) g^{(k)}(x) \frac{x^k}{k!}.
\end{aligned}
$$

Similarly, for the infinitely differentiable function $h(t)$, we can present

$$
\begin{aligned}
&g(xE)h(t)|_{t=0} = g(xe^D)h(t)\big|_{t=0} = \sum_{j=0}^{\infty} \frac{1}{j!} g^{(j)}(0) \left(xe^D\right)^j h(t)|_{t=0} \\
&= \sum_{j=0}^{\infty} \frac{x^j}{j!} g^{(j)}(0) \sum_{k=0}^{\infty} \frac{j^k}{k!} h^{(k)}(0) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} g^{(j)}(0) j^k \frac{x^j}{j!} \right) \frac{1}{k!} h^{(k)}(0).
\end{aligned}
$$

By applying (3.1) to the inner sum of the rightmost side of the above equation for $f(t) = t^k$ and noting $\left\{ \begin{matrix} k \\ j \end{matrix} \right\} = \left(\Delta^j t^k\right)_{t=0} / j!$, we obtain

$$
\begin{aligned}
&g(xE)h(t)|_{t=0} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \left(\Delta^j t^k\right)_{t=0} g^{(j)}(x) \frac{x^j}{j!} \right) \frac{1}{k!} h^{(k)}(0) \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} g^{(j)}(x) x^j \right) \frac{1}{k!} h^{(k)}(0) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} h^{(k)}(0) A_k(x, g(x)).
\end{aligned}
$$

This completes the proof of the theorem.

■

Remark 3.1 The series transformation formulas (3.1) and (3.2) could have numerous applications by setting different infinitely differentiable functions for $g(x)$. For examples, we have

$$\sum_{k=0}^{\infty} f(k)x^k = \sum_{k=0}^{\infty} \frac{x^k}{(1-x)^{k+1}}\Delta^k f(0) \quad (g(x)=(1-x)^{-1}), \tag{3.3}$$

$$\sum_{k=0}^{\infty} h(k)x^k = \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!}D^k h(0) \quad (g(x)=(1-x)^{-1}), \tag{3.4}$$

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} f(k)x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k}\frac{x^k}{(1+x)^{k-\alpha}}\Delta^k f(0) \quad (g(x)=(1+x)^{\alpha}), \tag{3.5}$$

$$\sum_{k=0}^{\infty} \binom{\alpha+k}{k} f(k)x^k = \sum_{k=0}^{\infty} \binom{\alpha+k}{k}\frac{x^k}{(1-x)^{\alpha+k+1}}\Delta^k f(0) \quad (g(x)=(1-x)^{-\alpha-1}), \tag{3.6}$$

$$\sum_{k=0}^{\infty} \frac{f(k)x^k}{k!} = e^x\sum_{k=0}^{\infty} \frac{x^k}{k!}\Delta^k f(0) \quad (g(x)=e^x), \tag{3.7}$$

$$\sum_{k=1}^{\infty} \frac{f(k)x^k}{k} = -f(0)\ln(1-x) + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x}{1-x}\right)^k \Delta^k f(0) \quad (g(x)=-\ln(1-x)), \tag{3.8}$$

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} h(k)x^k = \sum_{k=0}^{\infty} \frac{A_k(x,\alpha)}{k!}D^k h(0) \quad (g(x)=(1+x)^{\alpha}), \tag{3.9}$$

$$\sum_{k=0}^{\infty} \binom{\alpha+k}{k} h(k)x^k = \sum_{k=0}^{\infty} \frac{\tilde{A}_k(x,\alpha)}{k!}D^k h(0) \quad (g(x)=(1-x)^{-\alpha-1}), \tag{3.10}$$

$$\sum_{k=m}^{\infty} \binom{k}{m} f(k)x^k = \sum_{k=0}^{\infty} \binom{k+m}{m}\frac{x^{k+m}}{(1-x)^{k+m+1}}\Delta^k f(m) \quad (g(x)=(1-x)^{-m-1}), \tag{3.11}$$

$$\sum_{k=m}^{\infty} \binom{k}{m} h(k)x^k = \sum_{k=0}^{\infty} \frac{\tilde{A}_k(x,m)x^m}{k!}D^k h(m) \quad (g(x)=(1-x)^{-m-1}), \tag{3.12}$$

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k f(2k+1)x^{2k+1}}{(2k+1)!} &= \sin x\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}\Delta^{2k} f(0)}{(2k)!}\\ &\quad + \cos x\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}\Delta^{2k+1} f(0)}{(2k+1)!} \quad (g(x)=\sin x),\end{aligned} \tag{3.13}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k f(2k)x^{2k}}{(2k)!}=\cos x\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}\Delta^{2k}f(0)}{(2k)!}+\sin x\sum_{k=0}^{\infty}\frac{(-1)^{k+1}x^{2k+1}\Delta^{2k+1}f(0)}{(2k+1)!}\quad (g(x)=\cos x), \tag{3.14}$$

$$\sum_{k=0}^{\infty}\frac{f(2k)x^{2k}}{(2k)!}=\frac{e^x}{2}\sum_{k=0}^{\infty}\frac{x^k\Delta^k f(0)}{k!}+\frac{e^{-x}}{2}\sum_{k=0}^{\infty}\frac{(-x)^k\Delta^k f(0)}{k!}, \tag{3.15}$$

$$\sum_{k=0}^{\infty}\frac{f(2k+1)x^{2k+1}}{(2k+1)!}=\frac{e^x}{2}\sum_{k=0}^{\infty}\frac{x^k\Delta^k f(0)}{k!}-\frac{e^{-x}}{2}\sum_{k=0}^{\infty}\frac{(-x)^k\Delta^k f(0)}{k!}, \tag{3.16}$$

where (3.15) and (3.16) are obtained by replacing $g(x)$ by e^x and e^{-x} and adding and subtracting the resulting formulas respectively.

Note that (3.3)-(3.4) are well-known and have been utilized to construct summation formulas with estimable remainders. See, e.g., He-Hsu-Shiue-Torney [6]. The particular cases of (3.5) with $\alpha = m$ (positive integer) and (3.7) with $f(x)$ denoting a rth degree polynomial of x have been expounded in Problems (1109) and (1110) of Jolley's book [11]. The rest of the above list appears to be not easily found in literature, and the formulas (3.9)-(3.12) are believed to be new.

Apparently, (3.3) is implied by (3.5) (with $\alpha = -1$, $x \mapsto -x$) and (3.6) (with $\alpha = 0$). Also, (3.4) is a particular case of (3.9) (with $\alpha = -1$, $x \mapsto -x$) and (3.10) (with $\alpha = 0$). Moreover, it is easily observed that (3.11) and (3.12) can be derived from (3.6) and (3.10), respectively, by substituting $\alpha = m$, applying operator E^m, and multiplying x^m on the both sides of the former formulas.

The transformation formulas given in the list are useful for accelerating convergence of power series because $\Delta^k f(0)$ and $D^k f(0)$ decreases to zero rapidly as $k \to \infty$; e.g., the Euler series transformation and its extensions shown as in Proposition 3.2 of [6].

Remark 3.2 From (1.4) we can derive Gegenbauer type series transformation formulas. For examples, we consider

$$\Phi(t) = (1-2zt+t^2)^{-\lambda} = \sum_{k=0}^{\infty} C_k^{(\lambda)}(z)t^k,$$

the GF of $C_k^{(\lambda)}(z) \equiv P_k(2,z,1,\lambda)$, where $P_k(2,z,1,\lambda)$ was shown as in (1.4), and $C_k^{(1)}(z) = U_k(z)$ and $C_k^{(1/2)}(z) = \psi_k(z)$ are respectively the 2nd kind Chebyshev and Legengre polynomials. Using the same argument to derive (3.6) we obtain the following Gegenbauer type series transformation formula

$$\begin{aligned}
&\Phi(xE)f(0) = \sum_{k=0}^{\infty} C_k^{(\lambda)}(z)x^k f(k) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\lambda+i-1}{i}\binom{\lambda+j-1}{j}(z+\delta)^i(z-\delta)^j x^{i+j} f(i+j) \\
&= (1-2zx+x^2)^{-\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\lambda+i-1}{i}\binom{\lambda+j-1}{j} \\
&\quad \times \frac{(z+\delta)^i(z-\delta)^j x^{i+j}}{(1-z+\delta)x)^i(1-(z-\delta)x)^j}\Delta^{i+j}f(0). \qquad (3.17)
\end{aligned}$$

(3.17) can also be verified directly as follows. By denoting $\delta = \sqrt{z^2-1}$ we can expand $\Phi(xE)$ formal power series in terms of operator Δ as

$$\begin{aligned}
&\Phi(xE) = (1-(z+\delta)xE)^{-\lambda}(1-(z-\delta)xE)^{-\lambda} \\
= \;& (1-(z+\delta)x-(z+\delta)x\Delta)^{-\lambda}(1-(z-\delta)x-(z-\delta)x\Delta)^{-\lambda} \\
= \;& [1-(z+\delta)x]^{-\lambda}\left[1-\frac{(z+\delta)x\Delta}{1-(z+\delta)x}\right]^{-\lambda}[1-(z-\delta)x]^{-\lambda}\left[1-\frac{(z-\delta)x\Delta}{1-(z-\delta)x}\right]^{-\lambda} \\
= \;& (1-2zx+x^2)^{-\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\lambda+i-1}{i}\binom{\lambda+j-1}{j} \\
&\times \frac{(z+\delta)^i(z-\delta)^j x^{i+j}\Delta^{i+j}}{(1-z+\delta)x)^i(1-(z-\delta)x)^j}.
\end{aligned}$$

Thus, (3.17) is obtained.

In series transformation formula (3.17), we assume $f(t)$ to be a rth degree polynomial, denoted by $\phi(t)$, and obtain the generating function

$$GF\left\{C_k^{(\lambda)\phi(k)}(z)\right\}=\sum_{k=0}^{\infty}\left(C_k^{(\lambda)}(z)\phi(k)\right)x^k$$
$$=(1-2zx+x^2)^{-\lambda}\sum_{i=0}^{r}\sum_{j=0}^{r}\binom{\lambda+i-1}{i}\binom{\lambda+j-1}{j}$$
$$\times\frac{(z+\delta)^i(z-\delta)^jx^{i+j}\Delta^{i+j}\phi(0)}{(1-z+\delta)x)^i(1-(z-\delta)x)^j}. \tag{3.18}$$

In particular, for $\lambda=1$ and $1/2$ we have generating functions

$$GF\left\{\phi(k)U_k(z)\right\}=(1-2zx+x^2)^{-1}\sum_{i=0}^{r}\sum_{j=0}^{r}\binom{\lambda+i-1}{i}\binom{\lambda+j-1}{j}$$
$$\times\frac{(z+\delta)^i(z-\delta)^jx^{i+j}\Delta^{i+j}\phi(0)}{(1-z+\delta)x)^i(1-(z-\delta)x)^j},$$
$$GF\left\{\phi(k)\psi_k(z)\right\}=(1-2zx+x^2)^{-1/2}\sum_{i=0}^{r}\sum_{j=0}^{r}\binom{\lambda+i-1}{i}\binom{\lambda+j-1}{j}$$
$$\times\frac{(z+\delta)^i(z-\delta)^jx^{i+j}\Delta^{i+j}\phi(0)}{(1-z+\delta)x)^i(1-(z-\delta)x)^j}.$$

Remark 3.3 Evidently, when $f(t)$ is a polynomial, all the formulas in Section 3 become closed form of summation formulas with a finite number of terms. Moreover, the Right-hand side of each formula may also be viewed as a GF for the sequence of coefficeients contained in the power series on the left-hand side. Thus, for the rth degree polynomial $\phi(t)$, from (3.1) and (3.2) we obtain two type GF's of $\{\phi(k)g^{(k)}(0)\}$:

$$\sum_{k=0}^{\infty}\phi(k)g^{(k)}(0)\frac{x^k}{k!}=\sum_{k=0}^{r}\Delta^k\phi(0)g^{(k)}(x)\frac{x^k}{k!}, \tag{3.19}$$
$$\sum_{k=0}^{\infty}\phi(k)g^{(k)}(0)\frac{x^k}{k!}=\sum_{k=0}^{r}\frac{1}{k!}\phi^{(k)}(0)A_k(x,g(x)). \tag{3.20}$$

Replacing f and h by polynomial ϕ in (3.3)-(3.16), we obtain the special cases of (3.19) and (3.20). For instance,

$$\sum_{k=0}^{\infty}\phi(k)x^k=\sum_{k=0}^{r}\frac{x^k}{(1-x)^{k+1}}\Delta^k\phi(0)\quad (g(x)=(1-x)^{-1}),\tag{3.21}$$

$$\sum_{k=0}^{\infty}\phi(k)x^k=\sum_{k=0}^{r}\frac{\alpha_k(x)}{k!}D^k\phi(0)\quad (g(x)=(1-x)^{-1}),\tag{3.22}$$

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}\phi(k)x^k=\sum_{k=0}^{r}\binom{\alpha}{k}\frac{x^k}{(1+x)^{k-\alpha}}\Delta^k\phi(0)\quad (g(x)=(1+x)^{\alpha}),\tag{3.23}$$

$$\sum_{k=0}^{\infty}\binom{\alpha+k}{k}\phi(k)x^k=\sum_{k=0}^{r}\binom{\alpha+k}{k}\frac{x^k}{(1-x)^{\alpha+k+1}}\Delta^k\phi(0)\quad (g(x)=(1-x)^{-\alpha-1}),\tag{3.24}$$

$$\sum_{k=0}^{\infty}\frac{\phi(k)x^k}{k!}=e^x\sum_{k=0}^{r}\frac{x^k}{k!}\Delta^k\phi(0)\quad (g(x)=e^x),\tag{3.25}$$

$$\sum_{k=1}^{\infty}\frac{\phi(k)x^k}{k}=-f(0)\ln(1-x)+\sum_{k=1}^{r}\frac{1}{k}\left(\frac{x}{1-x}\right)^k\Delta^k\phi(0)\quad (g(x)=-\ln(1-x)),\tag{3.26}$$

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}\phi(k)x^k=\sum_{k=0}^{r}\frac{A_k(x,\alpha)}{k!}D^k\phi(0)\quad (g(x)=(1+x)^{\alpha}),\tag{3.27}$$

$$\sum_{k=0}^{\infty}\binom{\alpha+k}{k}\phi(k)x^k=\sum_{k=0}^{r}\frac{\tilde{A}_k(x,\alpha)}{k!}D^k\phi(0)\quad (g(x)=(1-x)^{-\alpha-1}),\tag{3.28}$$

$$\sum_{k=m}^{\infty}\binom{k}{m}\phi(k)x^k=\sum_{k=0}^{r}\binom{k+m}{m}\frac{x^{k+m}}{(1-x)^{k+m+1}}\Delta^k\phi(m)\quad (g(x)=(1-x)^{-m-1}),\tag{3.29}$$

$$\sum_{k=m}^{\infty}\binom{k}{m}\phi(k)x^k=\sum_{k=0}^{r}\frac{\tilde{A}_k(x,m)x^m}{k!}D^k\phi(m)\quad (g(x)=(1-x)^{-m-1}).\tag{3.30}$$

4 Illustrative examples

Certainly a great variety of special examples could be given via applications of the formulas displayed in Section 3. In what follows we merely present some selective examples for references.

Example 4.1 Taking $\alpha=-1$ and $x\mapsto -x$ in (3.5), we get (3.3), which is a well-known formula utilized in the constrction of a summation formula with a remainder in the recent paper by the authors [7] and the paper by authors with Torney (*cf.* [6]). Putting $x = -1$ in (3.3) and (3.8) we get

$$\begin{aligned}&\sum_{k=0}^{\infty}(-1)^k f(k) = \sum_{k=0}^{\infty}\frac{(-1)^k\Delta^k f(0)}{2^{k+1}}\\ &= \sum_{k=1}^{\infty}(-1)^k\frac{f(k)}{k} = -f(0)\log 2 + \sum_{k=1}^{\infty}\frac{(-1)^k\Delta^k f(0)}{k2^k}.\end{aligned}$$

These are known as Euler's series transform and its analogue, which may sometimes be used to construct slowly convergent series into rapidly convergent ones.

Example 4.2 From (3.5) and noting (2.1) and (2.2), we obtain the sum of the Euler's arithmetic-geometric series

$$\sum_{k=0}^{\infty}k^p x^k = \sum_{k=0}^{p}\frac{x^k\left[\Delta^k t^p\right]_{t=0}}{(1-x)^{k+1}} = \sum_{k=0}^{p}\frac{k!\left\{\begin{matrix}p\\k\end{matrix}\right\}x^k}{(1-x)^{k+1}} = \alpha_p(x),$$

where $\alpha_p(x)$ is known as Eulerian fraction (*cf.* [16]).

Example 4.3 In (3.5) taking $\alpha = n$, $f(t) = \binom{t}{j}$, a jth degree polynomial, so that $f(k) = \binom{k}{j}$, we get

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\binom{k}{j}x^k = \sum_{\nu=0}^{j}\binom{n}{\nu}\frac{x^\nu}{(1+x)^{\nu-n}}\Delta^\nu\binom{t}{j}_{t=0}\\ &= \sum_{\nu=0}^{j}\binom{n}{\nu}\frac{x^\nu}{(1+x)^{\nu-n}}\binom{t}{j-\nu}_{t=0} = \sum_{\nu=0}^{j}\binom{n}{\nu}\frac{x^\nu}{(1+x)^{\nu-n}}\delta_{j\nu}\\ &= \binom{n}{j}x^j(1+x)^{n-j},\end{aligned}$$

where we use $\Delta^k\binom{t}{r}_{t=0} = \binom{t}{r-k}_{t=0} = \binom{0}{r-k} = \delta_{rk}$, the Kronecker symbol. This is (3.118) of Gould's book [4].

Example 4.4 The series transformation formulas can be applied to construct a set of identities by substituting certain functions.

Similar to Example 4.3, taking $f(t) = \binom{t}{r}$ so that $f(k) = \binom{k}{r}$ in (3.6) yields (for $|x| < 1$)

$$\begin{aligned}\sum_{k=r}^{\infty} \binom{\alpha+k}{k}\binom{k}{r} x^k &= \sum_{k=0}^{r} \binom{\alpha+k}{k} \frac{x^k}{(1-x)^{\alpha+k+1}} \Delta^k \binom{t}{r}\bigg|_{t=0} \\ &= \binom{\alpha+r}{r} \frac{x^r}{(1-x)^{\alpha+r+1}}.\end{aligned}$$

Consequently,

$$\sum_{k=r}^{\infty} \binom{\alpha+k}{k}\binom{k}{r} \frac{1}{2^k} = \binom{\alpha+r}{r} 2^{\alpha+1}.$$

Similarly, for $f(t) = \binom{t}{r}$, $(r \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\})$, from (3.7)-(3.8) and (3.15)-(3.16) we obtain respectively

$$\begin{aligned}&\sum_{k=0}^{\infty} \binom{k}{r} \frac{x^k}{k!} = e^x \frac{x^r}{r!}, \\ &\sum_{k=1}^{\infty} \binom{k}{r} \frac{x^k}{k} = -\log(1-x) + \frac{1}{r}\left(\frac{x}{1-x}\right)^r \quad (r \geq 1), \\ &\sum_{k=0}^{\infty} \binom{2k}{r} \frac{x^{2k}}{(2k)!} = \frac{e^x}{2}\frac{x^r}{r!} + \frac{e^{-x}}{2}\frac{(-x)^r}{r!}, \\ &\sum_{k=0}^{\infty} \binom{2k+1}{r} \frac{x^{2k+1}}{(2k+1)!} = \frac{e^x}{2}\frac{x^r}{r!} - \frac{e^{-x}}{2}\frac{(-x)^r}{r!}.\end{aligned}$$

Example 4.5 In (3.6) taking $f(t) = t^r$ so that $f(k) = k^r$, we get

$$\sum_{k=0}^{\infty} \binom{\alpha+k}{k} k^r x^k = \sum_{k=0}^{r} \binom{\alpha+k}{k} \frac{x^k}{(1-x)^{\alpha+k+1}} k! \left\{ \begin{matrix} r \\ k \end{matrix} \right\}.$$

Formula (1.126) in [4] can be written as

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}k^r x^k &= (1+x)^n\sum_{j=0}^{r}\binom{n}{j}\frac{x^j}{(1+x)^j}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}k^r \\ &= \sum_{j=0}^{r}\binom{n}{j}\frac{x^j}{(1+x)^{j-n}}\left[\Delta^j t^r\right]_{t=0}.\end{aligned}$$

This is obviously a particular case of formula (3.5) with $\alpha = n$ and $f(t) = t^r$.

Example 4.6 Series transformation (3.11) can be used to extend Gould and Wetweerapong's comparable finite sum formula (*cf.* [5, (24)]) to the infinite sum setting, namely,

$$\sum_{k=0}^{\infty}\binom{k}{j}k^p x^k = \sum_{k=0}^{\infty}\binom{k+j}{j}\frac{x^{k+j}}{(1-x)^{k+j+1}}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(i+j)^p.$$

Example 4.7 It is known that the GF of Bell numbers $W(k)$ is

$$\sum_{k=0}^{\infty}W(k)\frac{x^k}{k!} = e^{e^x-1}.$$

Note that $W(k)$ is the number of all possible partition of a set with k distinct elements. Also, for $g(x) = e^x$ and $f(k) = W(k+1)$, formula (3.9) implies

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{k!}\Delta^k W(1)x^k = e^{-x}\sum_{k=0}^{\infty}\frac{1}{k!}W(k+1)x^k \\ = \; & e^{-x}\frac{d}{dx}\left(\sum_{k=0}^{\infty}\frac{1}{(k+1)!}W(k+1)x^{k+1}\right) = e^{-x}\frac{d}{dx}\left(e^{e^x-1}-1\right) \\ = \; & e^{e^x-1} = \sum_{k=0}^{\infty}\frac{1}{k!}W(k)x^k.\end{aligned}$$

Comparing the coefficients of x^k in the leftmost and the rightmost expressions, we get $W(k) = \Delta^k W(1)$, which is called the Aitken identity (*cf.* Theorem B in §5.4 of [1]).

Example 4.8 Our series transformation can be used to reconstruct the Dobinski's formula (*cf.* [4a] in §5.4 of [1]). If $g(t) = e^t$ and $f(t) = t^r$ ($r \in \mathbb{N}$), then (3.3) or (3.9) implies

$$\sum_{k=0}^{\infty} k^r \frac{x^k}{k!} = e^x \sum_{k=0}^{r} \left\{ \begin{array}{c} r \\ k \end{array} \right\} x^k.$$

This leads to

$$e^{-1} \sum_{k=0}^{\infty} \frac{k^r}{k!} = \sum_{k=0}^{r} \left\{ \begin{array}{c} r \\ k \end{array} \right\} = W(r).$$

Example 4.9 In (3.7)-(3.10) and (3.15)-(3.16), we substitute $f(t) = t^r$ and $h(t) = t^r$ and obtain respectively

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{k^r x^k}{k!} = e^x \sum_{k=0}^{r} \left\{ \begin{array}{c} r \\ k \end{array} \right\} x^k, \\
&\sum_{k=1}^{\infty} \frac{k^r x^k}{k} = -f(0)\log(1-x) + \sum_{k=1}^{r} (k-1)! \left\{ \begin{array}{c} r \\ k \end{array} \right\} \left(\frac{x}{1-x} \right)^k, \\
&\sum_{k=0}^{\infty} \binom{\alpha}{k} k^r x^k = A_r(x,\alpha)(r \in \mathbb{N} \cup \{0\}), \\
&\sum_{k=0}^{\infty} \binom{\alpha+k}{k} k^r x^k = \tilde{A}_r(x,\alpha)(r \in \mathbb{N}_0), \\
&\sum_{k=0}^{\infty} \frac{(2k)^r x^{2k}}{(2k)!} = \frac{e^x}{2} \sum_{k=0}^{r} \left\{ \begin{array}{c} r \\ k \end{array} \right\} x^k + \frac{e^{-x}}{2} \sum_{k=0}^{r} \left\{ \begin{array}{c} r \\ k \end{array} \right\} (-x)^k, \\
&\sum_{k=0}^{\infty} \frac{(2k+1)^r x^{2k+1}}{(2k+1)!} = \frac{e^x}{2} \sum_{k=0}^{r} \left\{ \begin{array}{c} r \\ k \end{array} \right\} x^k - \frac{e^{-x}}{2} \sum_{k=0}^{r} \left\{ \begin{array}{c} r \\ k \end{array} \right\} (-x)^k.
\end{aligned}$$

Example 4.10 In (3.5) taking $f(t) = r^t$, ($r > 0$, $r \neq 1$), so that $f(k) = r^k$ and $\Delta^k f(0) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} r^j = (r-1)^k$, we get

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} (rx)^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{((r-1)x)^k}{(1+x)^{k-\alpha}}.$$

Similarly, from (3.6)-(3.10) and (3.13)-(3.16) we have respectively

$$\sum_{k=0}^{\infty} \binom{\alpha+k}{k} (rx)^k = \sum_{k=0}^{\infty} \binom{\alpha+k}{k} \frac{((r-1)x)^k}{(1-x)^{\alpha+k+1}},$$
$$\sum_{k=0}^{\infty} \frac{(rx)^k}{k!} = e^x \sum_{k=0}^{\infty} \frac{((r-1)x)^k}{k!},$$
$$\sum_{k=1}^{\infty} \frac{(rx)^k}{k} = -f(0)\log(1-x) + \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{(r-1)x}{1-x}\right)^k,$$
$$\sum_{k=0}^{\infty} \binom{\alpha}{k} (rx)^k = \sum_{k=0}^{\infty} \frac{A_k(x,\alpha)}{k!} (\ln r)^k.$$

Example 4.11 Recall that Bernoulli polynomials $B_n(t)$'s are generated by the expression

$$e^{tx}\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^n$$

and enjoy the properties

$$\frac{d}{dt} B_n(t) = nB_{n-1}(t), \quad (n = 1, 2, \cdots)$$

with $B_0(t) = 1$ and $B_n(0) = B_n$ being called Bernoulli numbers. Note that $D^k B_n(t) = (n)_k B_{n-k}(t)$ so that $D^k B_n(0) = (n)_k B_{n-k}(0) = (n)_k B_{n-k}$, where $(n)_k$ are kth falling factorial of n with step length 1. Now let $g(x) = B_n(x)$ and $f(k) = k^r$ (n and r are integers with $0 \le r \le n$) Then, $g^{(k)}(0) = B_n^{(k)}(0) = (n)_k B_{n-k}$ and $f^{(k)}(x) = (n)_k B_{n-k}(x)$, so that Theorem 3.1 implies

$$\sum_{k=0}^{n} \binom{n}{k} B_{n-k} k^r x^k = \sum_{k=0}^{r} \binom{n}{k} B_{n-k}(x) k! \left\{ \begin{matrix} r \\ k \end{matrix} \right\} x^k. \tag{4.1}$$

Since $\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1$ and $\left\{ \begin{matrix} 0 \\ k \end{matrix} \right\} = 0$ for all $k \ge 1$, the particular case $r = 0$ gives the well-known expression

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

Of course the right-hand side of (4.1) can be regarded as a GF of $\{\binom{n}{k}B_{n-k}k^r\}_{k=0}^n$.

In addition, taking $f(t) = B_n(t)$ in (3.9) and (3.10) respectively, we easily obtain

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}B_n(k)x^k = \sum_{k=0}^{n}\binom{n}{k}A_k(x,\alpha)B_{n-k} \tag{4.2}$$

$$\sum_{k=0}^{\infty}\binom{\alpha+k}{k}B_n(k)x^k = \sum_{k=0}^{n}\binom{n}{k}\tilde{A}_k(x,\alpha)B_{n-k}. \tag{4.3}$$

Recalling that $\tilde{A}_k(x,0) = \alpha_k(x)$ (the ordinary Eulerian fraction), we can find the last identity implies (with $\alpha = 0$)

$$\sum_{k=0}^{\infty}B_n(k)x^k = \sum_{k=0}^{n}\binom{n}{k}\alpha_k(x)B_{n-k}. \tag{4.4}$$

Surely similar identities of some interest may be found for other classical special polynomials.

Example 4.12 Let λ and θ be any real numbers. The generalized falling factorial $(t+\lambda|\theta)_p$ is usually defined by

$$(t+\lambda|\theta)_p = \Pi_{j=0}^{p-1}(t+\lambda-j\theta),\ (p\geq 1),\ (t+\lambda|\theta)_0 = 1.$$

It is known that Howard's degenerate weighted Stirling numbers (*cf.* [9]) may be defined by the finite differences of $(t+\lambda|\theta)_p$ at $t=0$:

$$S(p,k,\lambda|\theta) := \frac{1}{k!}\left[\Delta^k(t+\lambda|\theta)_p\right]_{t=0}.$$

Then, using (3.23) and (3.24) with $\phi(t) = (t+\lambda|\theta)_p$, we get

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}(k+\lambda|\theta)_p x^k = \sum_{k=0}^{p}\binom{\alpha}{k}\frac{k!S(p,k,\lambda|\theta)x^k}{(1+x)^{k-\alpha}}, \tag{4.5}$$

$$\sum_{k=0}^{\infty}\binom{\alpha+k}{k}(k+\lambda|\theta)_p x^k = \sum_{k=0}^{p}\binom{\alpha+k}{k}\frac{k!S(p,k,\lambda|\theta)x^k}{(1-x)^{\alpha+k+1}}. \tag{4.6}$$

The particular case of (4.6) with $\alpha = 0$ was considered in [10]. It is also obvious that the classical Euler's summation formula for the arithmetic-geometric series (*cf.* for example, Lemma 2.7 in [5]) is implied by (4.5) with $\lambda = \theta = 0$, $\alpha = -1$, $x \mapsto -x$, or by (4.6) with $\lambda = \theta = 0$ and $\alpha = 0$.

Example 4.13 For any given positive integer m denote $\phi(t) = \binom{t}{m}$. It is easy to find that $\Delta^k \phi(m) = \binom{t}{m-k}_{t=m} = \binom{m}{k}$. Thus an application of (3.29) to $\binom{t}{m}$ gives

$$\sum_{k=m}^{\infty} \binom{k}{m}^2 x^k = \sum_{k=0}^{m} \binom{k+m}{k} \binom{m}{k} \frac{x^{k+m}}{(1-x)^{k+m+1}}. \tag{4.7}$$

This shows that the GF of the number sequence $\left\{\binom{k}{m}^2\right\}$ is given by

$$GF\left\{\binom{k}{m}^2\right\} = \sum_{k=0}^{m} \binom{k+m}{k} \binom{m}{k} \frac{x^{k+m}}{(1-x)^{k+m+1}}. \tag{4.8}$$

Naturally one may ask to find GF$\left\{\binom{k}{m}^3\right\}$. Actually, this can be worked out as follows.

Let the left-hand side of (4.7) be $\Phi(x)$. Then using (4.7) we find

$$\begin{aligned}
\Phi(xE)\phi(0) &= \sum_{k=m}^{\infty} \binom{k}{m}^2 (xE)^k \phi(0) = \sum_{k=m}^{\infty} \binom{k}{m}^3 x^k \\
&= \sum_{k=0}^{m} \binom{k+m}{k} \binom{m}{k} \frac{(xE)^{k+m}}{(1-xE)^{k+m+1}} \phi(0) \\
&= \sum_{k=0}^{m} \binom{k+m}{k} \binom{m}{k} \frac{x^{k+m}}{(1-x)^{k+m+1}} \left(1 - \frac{x\Delta}{1-x}\right)^{-k-m-1} E^{k+m}\phi(0) \\
&= \sum_{k=0}^{m} \binom{k+m}{k} \binom{m}{k} \frac{x^{k+m}}{(1-x)^{k+m+1}} \sum_{j=0}^{m} \binom{k+m}{j} \left(\frac{x}{1-x}\right)^j \Delta^j \phi(k+m) \\
&= \sum_{k=0}^{m} \sum_{j=0}^{m} \binom{k+m}{k} \binom{k+m}{j} \binom{m}{k} \binom{k+m}{k+j} \frac{x^{k+m+j}}{(1-x)^{k+m+j+1}}.
\end{aligned}$$

Thus we obtain

$$GF\left\{\binom{k}{m}^3\right\} =$$
$$\sum_{k=0}^{m}\sum_{j=0}^{m}\binom{k+m}{k}\binom{k+m}{j}\binom{m}{k}\binom{k+m}{k+j}\frac{x^{k+m+j}}{(1-x)^{k+m+j+1}}. \quad (4.9)$$

A similar process can be applied to find GF$\left\{\binom{k}{m}^n\right\}$ for $n = 4, 5, \cdots$. However, we have not yet known the closed form of $\sum_{k=0}^{\infty}\binom{k}{m}^{\ell} x^k$ for general ℓ.

Example 4.14 Suppose that $\phi(t)$ is an integral polynomial, namely, all its coefficients (including the constant term) are integers. It is easily seen that $\Delta^k\phi(0)/k!$ $(k = 0, 1, 2, \cdots)$ are integers as well. In fact, each term $a_m t^m$ $(m \geq 0)$ of $\phi(t)$ has a difference at zero: $\left[\Delta^k a_m t^m\right]_{t=0} = a_m k!\left\{\begin{matrix} m \\ k \end{matrix}\right\}$ with $\left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\} = 1$ and $\left\{\begin{matrix} m \\ k \end{matrix}\right\} = 0$ $(k > m)$. So $\Delta^k\phi(0)/k!$ is a linear combination of Stirling numbers of 2nd kind with integer coefficients. Thus formula (3.25) implies that $\sum_{k=0}^{\infty}\phi(k)x^k/k!$ is equal to e^x multiplying by an integral polynomial. In particular, for $x = 1$, this implies that

$$\frac{\phi(0)}{0!} + \frac{\phi(1)}{1!} + \frac{\phi(2)}{2!} + \cdots + \frac{\phi(k)}{k!} + \cdots$$

is an integral multiple of e.

Example 4.15 Every formula in Section 3 may be used to yield a pair of related formulas involving the trigonometric functions $\cos k\theta$ and $\sin k\theta$. For instance, setting $x = \rho e^{i\theta} = \rho(\cos\theta + \sin\theta)$ with $\rho = |x| > 0$ and $i^2 = -1$, we can obtain a pair of formulas from (3.24) as follows

$$\sum_{k=0}^{\infty} \binom{\alpha+k}{k} \phi(k)\rho^k \cos k\theta =$$
$$\sum_{k=0}^{r} \binom{\alpha+k}{k} \Delta^k \phi(0) Re \left(\frac{(\rho e^{i\theta})^k}{(1-\rho e^{i\theta})^{\alpha+k+1}} \right) \tag{4.10}$$
$$\sum_{k=0}^{\infty} \binom{\alpha+k}{k} \phi(k)\rho^k \sin k\theta =$$
$$\sum_{k=0}^{r} \binom{\alpha+k}{k} \Delta^k \phi(0) Im \left(\frac{(\rho e^{i\theta})^k}{(1-\rho e^{i\theta})^{\alpha+k+1}} \right), \tag{4.11}$$

where $Re(z)$ and $Im(z)$ denote respectively the real part and imaginary part of the complex number z. Obviously (4.10) and (4.11) could be specialized in various ways.

Remark 4.1 In this paper we have mostly considered the operator method for the cases when $\phi(t)$ takes various elementary functions. From Remarks 1.1, 3.2, and 3.4, we can see that the method also apply to the cases where $\phi(t)$ may take various suitable special functions. However, it still remains much to be investigated.

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