Review of counting techniques

- **Addition Principle**: Disjoint choices add

$$\text{If } A \cap B = \emptyset \text{ then } n(A \cup B) = n(A) + n(B)$$

- **Multiplication Principle**: Successive choices multiply:

$$n(A \times B) = n(A)n(B)$$

Taking k items from a set of n distinct things:

- If order matters and repetitions are allowed:

$$n^k$$

  by the multiplication principle.

- If order matters and repetitions are not allowed:

$$P_{n,k} = \frac{n!}{(n-k)!}$$

  also by the multiplication principle. **Permutations**

- If order does not matter and repetitions are not allowed:

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

  obtained by throwing away the order. **Combinations**

Situations Leading to Permutations

1. Anagrams
2. Lotteries in which winning tickets must match the order of the numbers drawn
3. Card games in which cards are dealt face up with play after each card is dealt
4. Election of officers
5. Assignment of people to distinct tasks

Situations Leading to Combinations

1. Card hands dealt all at once (most card games)
2. Picking out subsets of size k
3. Lotteries in which the numbers do not have to be matched in order
4. Choosing a committee of equals
5. Picking a work group to share a task

Situations leading to multiplication or powers

1. license plates
2. postal codes
3. phone numbers
4. repeated identical trials

Probability Basics

We set up an experiment which has several possible outcomes.

The **sample space** S is the set of all possible outcomes of the experiment.

An **event** is a subset of the sample space. $A \subseteq S$

A probability function assigns numbers to events such that

1. $P(A) \geq 0$
2. $P(S) = 1$
3. If $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$. More generally, if $\{A_\lambda | \lambda \in \Lambda\}$ is a disjoint countable set of events then $P(\bigcup A_\lambda) = \sum P(A_\lambda)$.

Note that we give probabilities for **events** (subsets of S) not for **outcomes** (elements of S).
In general there are a lot of events. If S has n elements, then there are 2^n possible subsets. If n is large, that will be huge.
Our object is to figure out how to calculate probabilities from knowing only some of the probabilities of events.

We know that A and its complement A' are disjoint and have $A \cup A' = S$, so

$$P(A) + P(A') = P(S) = 1$$

Thus to find the probability of an event it often pays to find the probability of its complement.

Example

Suppose I pick two students at random. What is the probability they have different birthdays?

There is only a probability of $1/365$ that the second student has the same birthday as the first, so the probability that the birthdays are different is $364/365$.

We also know that we can decompose $A \cup B$ into disjoint pieces as

$$A \cup B = (A \cap B') \cup (A \cap B) \cup (A' \cap B)$$

so

$$P(A \cup B) = P(A \cap B') + P(A \cap B) + P(A' \cap B)$$

and in general

$$P(A \cup B) = P(A) - P(A \cap B) + P(B).$$

For finite sample spaces one approach is to give the probabilities for simple events. Simple events have the form $\{s_i\}$ where $s_i \in S$. Taking the singleton set makes this an event rather than an outcome.

In order for the conditions on a probability function to hold we need each $0 \leq P(\{s_i\}) \leq 1$ and $\sum_S P(\{s_i\}) = 1$.

You can then find the probability of an event A by adding up the probabilities of the simple events which are subsets of A.

$$P(A) = \sum_{a \in A} P(\{a\})$$

Equiprobable spaces

In many situations (like deals from a well shuffled deck or rolls of a fair die or tosses of a fair coin) each of the simple events has the same probability.

$$P(\{a\}) = \frac{1}{n(S)}$$

In this case (and only in this situation)

$$P(A) = \frac{n(A)}{n(S)}$$

which makes probabilities easy to calculate in these situations, provided we can count.

Example

If you deal 5 cards from a well shuffled deck what is the probability that you get 4 aces?

Here we count. The sample space consists of all of the 5 card hands. there are $C_{52,5}$ of them. If we take all four aces and one non-ace there will be 48 ways to choose the hand. Thus the probability is

$$\frac{48}{\frac{52!}{5!47!}} = \frac{1}{54145}.$$

Example

Roll two fair dice. The outcomes are in the following table, each equally probable:

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
5	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

There are 36 possible outcomes.

Example (continued)

If we just keep track of the sum the results are

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
5	7	8	9	10	11	12

Some probabilities just from counting:

- $P(\text{the sum is } 5) = 4/36 = 1/9$
- $P(\text{the sum is a multiple of } 3) = 12/36 = 1/3$
- $P(\text{the sum is a multiple of } 5) = 7/36$
- $P(\text{the sum is a } 7 \text{ or } 11) = 8/36 = 2/9$
- $P(\text{the dice match}) = 6/36 = 1/6$

Example

A club has 100 members, of whom 63 are women. An executive committee is chosen with 5 members, all of whom are men. What is the probability of this happening if the committee members were chosen randomly?

We can choose the committee $C_{100,5}$ ways. We can choose a committee of all men in $C_{37,5}$ ways, so the probability is

$$\frac{\left(\frac{37!}{5!32!}\right)}{\left(\frac{100!}{5!95!}\right)} = \frac{629}{108640} \approx 0.00578976$$

Example (continued)

What is the probability that our committee has exactly 2 men?
Here we again can count:

$$P(\text{exactly 2 men on 5 person committee}) = \frac{\left(\frac{37!}{2!35!}\,\frac{63!}{3!60!}\right)}{\left(\frac{100!}{5!95!}\right)}$$

$$= \frac{209901}{597520} \approx 0.351287$$