

# Basic Mathematics of Finance

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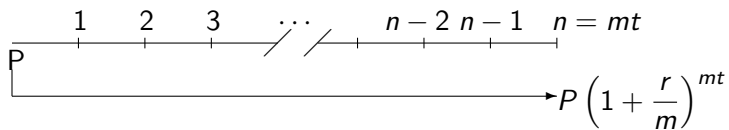
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Symbol	meaning and mnemonic
$P$	<u>P</u> rinciple
$r$	annual interest <u>r</u> ate
$t$	number of years, <u>t</u> ime
$m$	number of <u>co</u> mpounding periods per year
$mt$	number of compounding periods in $t$ years
$n$	number of compounding or payment periods $n = mt$
$i$	<u>I</u> nterest rate per compounding period $i = \frac{r}{m}$
$P_n$	future value after $n$ periods
$R$	payment amount made at a <u>r</u> egular interval
$A$	present value of an <u>a</u> nnuity
$S$	future value of an annuity <u>S</u> avings plan or <u>S</u> inking fund

We find the amount which accumulates when a principle  $P$  is put in an account with an annual interest rate  $r$  compounded  $m$  times a year for  $t$  years is given by

$$P_n = P \left( 1 + \frac{r}{m} \right)^{mt}$$



## Example

Suppose I deposit \$2000 now at 5% annual interest compounded monthly and ask how much I will have in that account 15 years from now. Here  $P = \$2000$ ,  $r = .05$ ,  $m = 12$ , and  $t = 15$ . I will end up with

$$\$2000 \left( 1 + \frac{.05}{12} \right)^{12 \cdot 15} = \$4227.41$$

Doing the same calculation for someone who needs the money in 45 years instead gives

$$\$2000 \left( 1 + \frac{.05}{12} \right)^{12 \cdot 45} = \$18,886.98$$

With compound interest it pays to wait longer!

## Example

A related calculation lets me calculate how much I would need to deposit now to have \$1,000,000 in 15 years, again assuming 5% annual interest compounded monthly. Here I take

$$1000000 = P \left( 1 + \frac{.05}{12} \right)^{12 \cdot 15}$$

and solve for  $P$ :

$$P = \frac{1000000}{\left( 1 + \frac{.05}{12} \right)^{12 \cdot 15}} = 473103.16$$

## Example

We can also see how long it takes my \$2000 to grow into \$10000 by solving

$$10000 = 2000 \left(1 + \frac{.05}{12}\right)^{12 \cdot t}$$

for  $t$  using logarithms:

$$5 = \frac{10000}{2000} = \left(1 + \frac{.05}{12}\right)^{12 \cdot t}$$

$$\log(5) = 12t \log\left(1 + \frac{.05}{12}\right)$$

$$\frac{\log(5)}{12 \log\left(1 + \frac{.05}{12}\right)} = t$$

$$t = 32.26 \text{ years}$$

An annuity is a regular sequence of payments. We usually assume that the compounding period and the payment period coincide, so monthly payments go with monthly compounding. We will consider an *annuity due*: regular payments of size  $R$  are made at the end of each period for  $n$  periods. We assume an interest rate of  $i$  per period.



When working with annuities we will need the formula for the sum of a geometric series:

If

$$S = a_0 + a_0r + a_0r^2 + \dots + a_0r^n$$

then

$$rS = a_0r + a_0r^2 + \dots + a_0r^n + a_0r^{n+1}$$

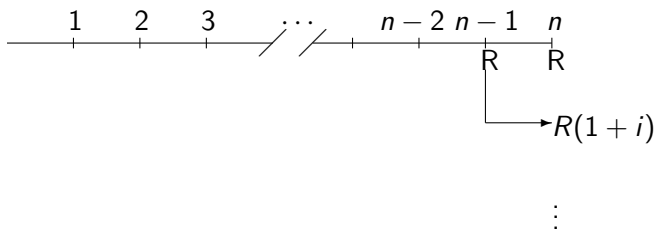
Subtracting gives

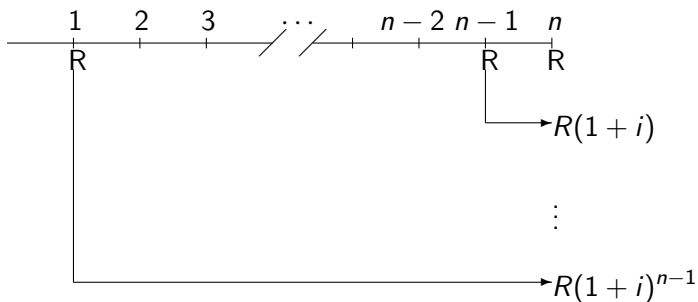
$$(1 - r)S = a_0(1 - r^{n+1}) \text{ so } S = \frac{a_0(1 - r^{n+1})}{1 - r}$$

To find the value of this sequence of payments right after the  $n^{\text{th}}$  payment is made we find the future value of each of the payments:









So the sum of the present values of the payments is

$$R + R(1 + i) + R(1 + i)^2 + R(1 + i)^3 + \cdots + R(1 + i)^{n-1}$$

a geometric series with  $n$  terms, common ratio  $1 + i$ , and first term  $R$ . The formula for the sum of a geometric series gives

$$S = R \frac{1 - (1 + i)^n}{1 - (1 + i)} = R \frac{1 - (1 + i)^n}{-i} = R \frac{(1 + i)^n - 1}{i}$$

## Example

Let us consider the future value of \$1000 paid at the end of each month into an account paying 8% annual interest for 30 years. How much will accumulate? This is a future value calculation with  $R = 1000$ ,  $n = 360$ , and  $i = \frac{.08}{12}$ . This account will accumulate

$$\$1000 \frac{(1 + \frac{.08}{12})^{360} - 1}{\frac{.08}{12}} \approx \$1,490,359.45$$

Note that this is much larger than the sum of the payments, since many of those payments are earning interest for many years.

We can also figure out what payment would need to be made each month to achieve a financial goal.

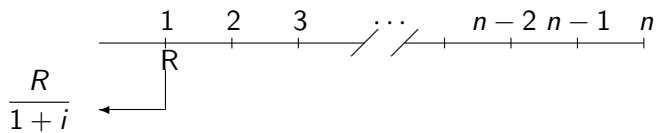
### Example

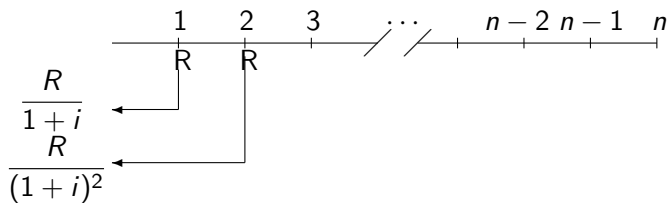
How much do you have to resolve to save at the end of each month in order to accumulate \$24000 in 4 years if you can get 6% annual interest compounded monthly? Here we know  $S = 24000$ ,  $n = 48$  and  $i = \frac{.06}{12}$ , and we solve for  $R$ :

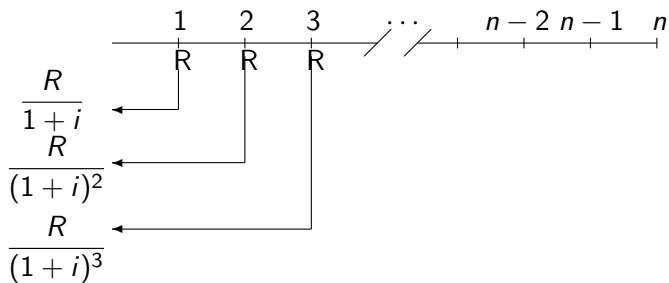
$$\begin{aligned} \$24000 &= R \frac{(1.005)^{48} - 1}{.005} \\ R &= \$24000 \frac{.005}{(1.005)^{48} - 1} \\ &= \$443.64 \end{aligned}$$

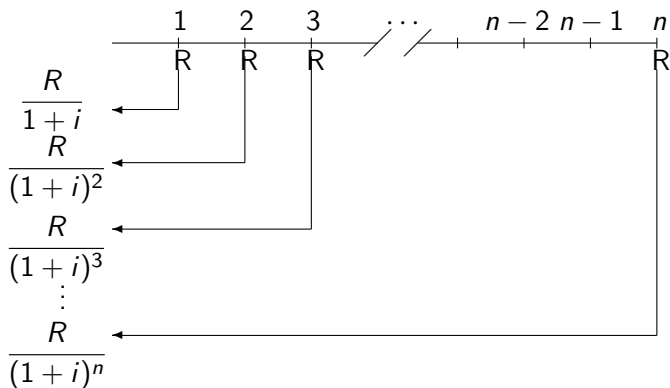
Here the total of the payments is  $\$443.64 \cdot 48 = \$21294.72$  the rest comes from the interest earned.

Sometimes we want to know how much the annuity is worth at the beginning of the picture. Here we move each of the payments back in time:









So the sum of the present values of the payments is

$$\frac{R}{1+i} + \frac{R}{(1+i)^2} + \frac{R}{(1+i)^3} + \cdots + \frac{R}{(1+i)^n}$$

a geometric series with  $n$  terms, common ratio  $\frac{1}{1+i}$ , and first term  $\frac{R}{1+i}$ .  
The formula for the sum of a geometric series gives

$$A = \left( \frac{R}{1+i} \right) \frac{1 - \left( \frac{1}{1+i} \right)^n}{1 - \frac{1}{1+i}} = R \frac{1 - (1+i)^{-n}}{i}$$

## Example

A typical calculation asks what size loan you could get if you can afford to pay \$1000 per month for 30 years at 8% annual interest: here

$R = 1000$ ,  $i = \frac{.08}{12}$ , and  $n = 12 \cdot 30 = 360$ . Thus the amount you could borrow is

$$\$1000 \frac{1 - \left(1 + \frac{.08}{12}\right)^{-360}}{\frac{.08}{12}} \approx 136283.49$$

Under these terms you would end up paying a total of \$360,000, so the total interest paid would be  $\$360,000 - \$136,283.49 = \$223,716.51$ .

Often a payment is considered to consist of a portion which pays the interest on the outstanding balance of the loan for one period and a portion which reduces the balance due. Such calculations are needed for finding home interest deductions on taxes, for instance.

### Example

Payments of \$1000 per month are being made on a 30 year mortgage at 8% annual interest (as in the previous example). What is the balance due right after the payment 40 has been made? Since there were 360 payments in the original loan and 40 of them have been made there are 320 left. So we should figure out what the loan would be which would result in 320 monthly payments of \$1000 using 8% annual interest:

$$\$1000 \frac{1 - \left(1 + \frac{.08}{12}\right)^{-320}}{\frac{.08}{12}} \approx \$132107.50$$

The interest portion of the payment 41 is then one month's interest on that outstanding balance, or

We can also figure out what the payments will be for a loan of a particular size:

### Example

If you decide to borrow \$150000 under these same terms, then the payment will satisfy

$$R \frac{1 - \left(1 + \frac{.08}{12}\right)^{-360}}{\frac{.08}{12}} = 150000$$

so that

$$R = \frac{150000}{\frac{1 - \left(1 + \frac{.08}{12}\right)^{-360}}{\frac{.08}{12}}} \approx 1100.65$$

Solving for either the interest rate or the number of payments is somewhat more difficult.

### Example

If you borrow \$1000 at 8% compounded monthly and agree to pay it off with \$10 per month payments, how long will it take to pay off the debt? Note that one month's interest on \$1000 is \$6.67, so this could easily take close to 200 months. To find an exact answer we solve

$$1000 = 10 \frac{1 - \left(1 + \frac{.08}{12}\right)^n}{\frac{.08}{12}}$$

for  $n$ :

$$\begin{aligned} 1000 &= 10 \frac{1 - \left(1 + \frac{.08}{12}\right)^{-n}}{\frac{.08}{12}} \\ 100 &= \frac{1 - \left(1 + \frac{.08}{12}\right)^{-n}}{\frac{.08}{12}} \end{aligned}$$

## Example (continued)

$$\begin{aligned}\frac{8}{12} &= 1 - \left(1 + \frac{.08}{12}\right)^{-n} \\ \left(1 + \frac{.08}{12}\right)^{-n} &= 1 - \frac{8}{12} \\ -n \log \left(1 + \frac{.08}{12}\right) &= \log\left(1 - \frac{8}{12}\right) \\ n &= -\frac{\log\left(1 - \frac{8}{12}\right)}{\log\left(1 + \frac{.08}{12}\right)} \\ &= 165.34\end{aligned}$$

Under these circumstances the final payment (the 166<sup>st</sup>) would be recalculated before it was made, since it will be less than the regular \$10.