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3. **The Division Principle** If B is divided up into disjoint subsets which are the same size as A then the number of such subsets is $n(B)/n(A)$ Sometimes it pays to make a *systematic* overcount.

Example

American zip codes consist of 6 digits, so there are $10^6 = 1,000,000$ of them.
Canadian postal codes consist of 3 letters and 3 digits (for example H3C 3G1, the postal code for McGill University). There are $26^3 10^3 = 17,576,000$ of them.

Example (Permutations of n things k at a time)

If we have a set of n different items and we want to know how many ways we can choose k of them in order then we have a succession of choices with one less option each time:

$$\begin{aligned} P_{n,k} &= n \times (n-1) \times \ldots \times (n-k+1) \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

Example

A club has ten members. How many possible ways are there for it to elect a president, vice-president, secretary, and treasurer?

We assume that no club member will hold two offices and that it makes a difference which office is held. This makes this a permutations example.

We want

$$P_{10,4} = \frac{10!}{6!} = 10 \times 9 \times 8 \times 7 = 5040$$

Another situation giving rise to permutations is finding "words" of a given length using tiles with letters on them making up some given word:

Example

How many 4 letter anagrams are there taken from the word **NORMAL**? Here we want

$$\begin{aligned} P_{6,4} &= \frac{6!}{2!} \\ &= 6 \times 5 \times 4 \times 3 \\ &= 360 \end{aligned}$$

If we have repeated letters we will need to throw out the order on those letters. This is a use of the division principle:

Example

How many distinct ways are there to rearrange the letters in **ILLINOIS**?
There are 8 letters: 3 I's, 2 L's, and one each of N, O, and S.
The number we are looking for is:

$$\frac{8!}{3!2!} = 3360$$

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I			L			N, O, S	
1	$\frac{7!}{2!2!}$	$+$	1	$\frac{7!}{3!}$	$+$	3	$\frac{7!}{3!2!}$

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$$\begin{array}{ccccccccc} \text{I} & & & \text{L} & & & \text{N, O, S} & \\ 1 & \frac{7!}{2!2!} & + & 1 & \frac{7!}{3!} & + & 3 & \frac{7!}{3!2!} \end{array}$$

$$= 3360$$

If we want to count how many ways there are to choose a subset with k elements from a set of n elements we start by choosing ordered k-tuples and then throw out the order. This gives the number of **combinations** of n things k at a time:

$$C_{n,k} = \left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$$

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Example

Pick three numbers from the numbers from 1 to 40. They don't have to be different. You win the grand prize if your numbers match the numbers drawn in exactly the same order. How many ways can you pick numbers for this lottery?

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There are $40^3 = 64000$ possible tickets.

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Another game lets you pick 4 numbers between 1 and 44, but they all need to be different. Four balls with numbers on them are then taken all at once from a drum. You win if the set of numbers drawn matches your picks. How many ways can you make your choices?

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Here we are choosing combinations of 44 things 4 at a time, so we get

$$\frac{44!}{4!40!} = 135751$$

Combinations often come up when counting card hands:

Example

How many 5 card hands are there from a standard deck of 52?
Order doesn't matter in a card hand if all cards are dealt at once: That's the number of combinations of 52 things 5 at a time:

$$C_{52,5} = \frac{52!}{5!47!} = 2,598,960$$

Example (continued)

How many of those hands have 3 aces?
First pick the aces, then pick the non-aces:

$$\frac{4!}{1!3!} \cdot \frac{48!}{2!46!} = 4512$$

Example (continued)

How many of the five card hands have 2 pairs?
First pick the kinds for the two pairs ($C_{13,2}$), then pick the cards in those types ($C_{4,2}$ for each), then pick the remaining card:

$$\frac{13!}{2!11!} \cdot \frac{4!}{2!2!} \cdot \frac{4!}{2!2!} \cdot 44 = 123552$$

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$$2 \cdot 30 \cdot 70 \cdot \frac{69!}{8!61!} + 70 \cdot 69 \cdot \frac{68!}{7!61!} 30 = 175,590,527,112,000$$