Essay Assignment on the δ-ϵ Definition of a Limit

One of the key concepts in calculus is the concept of the limit of a function, $\lim_{x \to a} f(x) = L$ Consideration of simple examples and counterexamples suggests that we need to exclude the point a from consideration and that we want values of $f(x)$ to get arbitrarily close to L when the value of x gets close enough to a. We expect polynomials to be well behaved with respect to limits, and that limits will play well with operations on functions but anticipate some difficulties may arise from division by a function approaching zero. While this intuition is useful, we will need a more precise definition to *prove* any theorems and nail down the examples.

One of the characteristics of mathematics since the ancient Greeks has been precision of definitions and proof of assertions. Without carefully crafted definitions proof of theorems becomes impossible. But carefully crafted definitions, like the δ-ϵ definition of a limit, are subtle. You need to learn how to decipher a definition in order to succeed in mathematics. The object of this assignment is for you to learn how to decipher a definition while you learn the δ-ϵ definition of a limit.

I want you to write an essay in which you explain in detail the definition of a limit. This will take about three to five pages. I would prefer that it came in typed. Write it in good English prose with coherent paragraphs. These will be due Monday, September 13, 2010. I will read them, comment on them, and assign up to 20 points using a rubric based on what this assignment sheet calls for. This is, like all other homework assignments in this course, intended to help you learn calculus.

Calculus didn't have a rigorous definition of a limit for the first century of its existence. A long search for a way to make calculus rigorous followed a shift from basing rigor on Euclidean geometry to basing rigor on an algebraic description of numbers. A version of the definition of a limit developed by Cauchy and Weierstrass in the 1820's uses this arithmetization of analysis and is given by

Definition If f is defined for all values close enough to a, though not necessarily at a, then $\lim_{x \to a} f(x) = L$ means that for every $\epsilon > 0$ there is a $\delta > 0$ such that for any x if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

To understand a definition you must first decipher what all of its parts mean. To this end you should include in your essay explanations of what the inequalities $0 < |x-a|$, $|x-a| < \delta$, and $|f(x) - L| < \epsilon$ tell you about x and $f(x)$. Why do you need $0 < |x - a|$? Why don't you want to insist on $0 < |f(x) - L|$?

The next thing you need to do to understand a definition is to figure out how the parts fit together. In this definition you need to start by understanding the logical structure of the definition. Look out for the little words: *for all, there exists, if, then.* Which of the numbers δ and ϵ depends on the other? What direction does the argument connecting $0 < |x-a| < \delta$ and $|f(x) - L| < \epsilon$ go? What precisely do you have to do to show that a given function has a given limit at a given point?

You don't really understand a definition without examples, so your essay should include at least two examples showing how different situations can be and still meet the definition and an example or two showing how the definition can fail. If possible, you want to show how the definition applies in each of these examples. I would recommend avoiding complicated limits in your examples as the technical difficulties can obscure what is going on. At least one of your examples should have the function undefined at a.

One of the most important tests of a definition (after checking examples to see if it captures the desired notion) is the ease with which it can be used to prove theorems. This is what we will be doing in class as you are finishing this paper.